In these notes, we will write out some basics of Arakelov geometry in the 1-dimensional case. We follow both [Mor14] and these notes by Chambert-Loir.

1 Basic Notions

Let K be a number field, $R = \mathcal{O}_K$ the ring of integers, $S = \mathrm{Spec}(R)$, $K(\mathbf{C})$ the set of embeddings of K in $\mathbf{C}$. It is also possible to do all the theory below over an order in R (a $\mathbf{Z}$-lattice of K inside R, geometrically a singular integral curve with normalization S), as [Mor14] does, but we will not do this—I am happy to work over a Dedekind domain so everything is easier.

Definition 1.1. The group of *arithmetic divisors* on R is $Z^1(R) \oplus \left(\bigoplus_{\sigma \in K(\mathbf{C})} \mathbf{R}\sigma\right)$, where $Z^1(R)$ is the usual group of divisors on R. We denote this group by $\widehat{Z}^1(R)$ of $\widehat{Z}^1(R)$. A typical element will be written as (D, g), where D is the finite part and g is the infinite part.

In general, our notation will add hats for the arithmetic analogues of classical objects. The idea is that we manually input the archimedean data, which lies over the (nonexistent) points of the affine curve S that would compactify it. In this way we can make arguments that mirror the ones we would make in the geometric case, where S would be a smooth projective/proper curve over a field. This is basically the philosophy of Arakelov theory.

Definition 1.2. The *degree* of an arithmetic divisor (D, g), where $D = \sum_{\mathfrak{p} \in \mathrm{maxSpec}(\mathcal{O}_K)} n_\mathfrak{p} \mathfrak{p}$, is

$$\widehat{\deg}(D, g) = \sum_{\mathfrak{p} \in \mathrm{maxSpec}(\mathcal{O}_K)} n_\mathfrak{p} \log N(\mathfrak{p}) + \frac{1}{2} \sum_{\sigma \in \Sigma} g_\sigma.$$

This is a homomorphism $\widehat{Z}^1(R) \to \mathbf{R}$.

Remark 1.3. Note that since we are no longer working over a base field k, we no longer have notion of dimension of a vector space. Therefore $\log N(\mathfrak{p})$ plays this role, as an analogue of $\dim_{\mathbf{F}_p}(R/\mathfrak{p}) = \log_p [R/\mathfrak{p} : \mathbf{F}_p]$.

Recall that in the classical case, for $a \in K^*$, there should be some natural construction of an associated (arithmetic) divisor $\widehat{\mathrm{div}}(a)$ with degree 0.

Definition 1.4. Given $a \in K^*$, we define

$$\widehat{\mathrm{div}}(a) = (\mathrm{div}(a), (-2 \log|\sigma(a)|)_{\sigma \in \Sigma})$$

where $\mathrm{div}(a) = \sum_{\mathfrak{p} \in \mathrm{maxSpec}(\mathcal{O}_K)} \mathrm{ord}_\mathfrak{p}(a) \mathfrak{p}$.

Remark 1.5. The normalizations of 1/2 in Definition 1.7 and 2 in Definition 1.4 are there since we are using *all* embeddings of K into $\mathbf{C}$, and not just half of the complex conjugate embeddings (one from each conjugate pair of complex embeddings). This differs from the conventions of Chambert–Loir.

Proposition 1.6. For any $a \in K^*$, $\widehat{\deg}(\widehat{\operatorname{div}}(a)) = 0$.

Proof. Taking the sum over primes $\mathfrak{p}$ containing a, we have

$$\widehat{\deg}(\widehat{\operatorname{div}}(a)) = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(a)\log(N(\mathfrak{p})) - \sum_{\sigma\in\Sigma}\log|\sigma(a)| = \log\prod_{\mathfrak{p}} N(\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}}(a)} - \log\prod_{\sigma\in\Sigma}|\sigma(a)|$$
$$= \log N(aR) - \log N_{K/\mathbf{Q}}(a).$$

But by the theory of modules over a PID, the last number is 0 (the first term being the size of the cokernel of the $\mathbf{Q}$-linear map $K \to K$ given by multiplication by a, the second term being the determinant of that map). □

Definition 1.7. We call the quotient $\widehat{CH}^1(R) \coloneqq \widehat{Z}^1(R)/\widehat{\operatorname{Rat}}^1(R)$ the *first arithmetic Chow group* of S. By the previous proposition, $\widehat{\deg}$ induces a homomorphism $\widehat{CH}^1(R) \to \mathbf{R}$.

Example 1.8. When $K = \mathbf{Q}$, $\widehat{\deg} : \widehat{CH}^1(\operatorname{Spec}(\mathbf{Z})) \to \mathbf{R}$ is an isomorphism. Injectivity follows from the fact that $\mathbf{Q}$ is a PID.

We now turn to the "geometric" side of things. For now, we treat things in slightly more generality than perhaps necessary. Let H be a finitely generated R-module. For $\sigma \in K(\mathbf{C})$, we denote by H_σ the tensor $H \otimes_{R \xrightarrow{\sigma} \mathbf{C}} \mathbf{C}$.

Proposition 1.9. The natural $\mathbf{C}$-linear homomorphism $\phi : H \otimes_{\mathbf{Z}} \mathbf{C} \to \bigoplus_{\sigma\in K(\mathbf{C})} H_\sigma$ given by $h \otimes \alpha \mapsto (h \otimes \alpha)_\sigma$ is bijective. Moreover, if we set $F_\infty : H \otimes_{\mathbf{Z}} \mathbf{C} \to H \otimes_{\mathbf{Z}} \mathbf{C}$ by $F_\infty(h \otimes \alpha) = h \otimes \overline{\alpha}$, and $F'_\infty : \bigoplus_{\sigma\in K(\mathbf{C})} H_\sigma \to \bigoplus_{\sigma\in K(\mathbf{C})} H_\sigma$ as induced by the natural anti-$\mathbf{C}$-linear maps $H_\sigma \to H_{\overline{\sigma}}$ ($h \otimes \alpha \mapsto h \otimes \overline{\alpha}$), then $\phi \circ F_\infty = F'_\infty \circ \phi$. In particular, $\phi(H \otimes_{\mathbf{Z}} \mathbf{R})$ is the set of $x \in \bigoplus_{\sigma\in K(\mathbf{C})} H_\sigma$ such that $F'_\infty(x) = x$.

Proof. It suffices to prove the first assertion in the case $H = R$, because then

$$H \otimes_R (R \otimes_{\mathbf{Z}} \mathbf{C}) \to H \otimes_R \left(\bigoplus_{\sigma\in K(\mathbf{C})} R_\sigma\right)$$

is bijective, and unraveling these maps shows that ϕ is bijective. So we reduce to the case $H = R$. Noting that $R \otimes_{\mathbf{Z}} \mathbf{C} = K \otimes_{\mathbf{Q}} \mathbf{C}$ (naturally) and $R \otimes_R^\sigma \mathbf{C} = K \otimes_K^\sigma \mathbf{C}$ for all $\sigma \in K(\mathbf{C})$, we just need to show that $K \otimes_{\mathbf{Q}} \mathbf{C} \to \bigoplus_{\sigma\in K(\mathbf{C})} K_\sigma$ is bijective. This is well-known to be bijective, upon writing $K/\mathbf{Q}$ as a simple extension $\mathbf{Q}(r)$ and considering this map as $f(r) \otimes \alpha \mapsto (f(\sigma(r))\alpha) = (\sigma(f(r))\alpha)$. Finally, given this bijection, $\phi \circ F_\infty = F'_\infty \circ \phi$ is obvious, as is the last statement. □

Next, suppose we equip each H_σ with a Hermitian inner product h_σ. We call the data $\overline{H} := (H, h) := (H, (h_\sigma))$ a *Hermitian R-module.*

Definition 1.10. We say a Hermitian R-module (H, h) is *of real type* if $h_{\overline{\sigma}}(x \otimes^{\overline{\sigma}} 1, y \otimes^{\overline{\sigma}} 1) = \overline{h_\sigma(x \otimes^\sigma 1, y \otimes^\sigma 1)}$ holds for all σ and $x, y \in H$. This expresses the "conjugation invariance" of the family h of metrics.

Example 1.11. The canonical Hermitian metric h^{can} on R (rank 1) given by $h^{can}_\sigma(x \otimes 1, y \otimes 1) = \sigma(x)\overline{\sigma(y)}$ is clearly of real type.

Now for a Hermitian R-module H, we set $\langle x, y \rangle_h = \sum_{\sigma \in K(\mathbf{C})} h_\sigma(x \otimes 1, y \otimes 1)$. Note that this is in $\mathbf{R}$ if (H, h) is of real type (conjugation-invariant). Then

Proposition 1.12. The above pairing gives a positive definite Hermitian form on H, by which we mean it is a antisymmetric $\mathbf{Z}$-bilinear form on H that becomes a positive definite inner product on $H \otimes_{\mathbf{Z}} \mathbf{C}$ (or $H \otimes_{\mathbf{Z}} \mathbf{R}$, if h is of real type) via $\langle x \otimes a, y \otimes b \rangle := a\bar{b}\langle x, y \rangle_h$ (or the form is identically 0, as is the case when H is torsion as a $\mathbf{Z}$-module).

Proof. It is easy to verify that the above is a Hermitian form. To show the positive-definiteness, use the isomorphism of Proposition 1.9. The $\mathbf{C}$-vector space $\bigoplus_{\sigma \in K(\mathbf{C})} H_\sigma$ has a Hermitian inner product given by the direct sum of the inner products h_σ (with different components defined to be mutually orthogonal), i.e.

$$h'((x_\sigma), (y_\sigma)) = \sum_\sigma h_\sigma(x_\sigma, y_\sigma).$$

In particular this is positive definite. Transporting this inner product over the isomorphism of Proposition 1.9 gives an inner product on $H \otimes_{\mathbf{Z}} \mathbf{C}$ via

$$\langle x \otimes a, y \otimes b \rangle' = \sum_\sigma a\bar{b}h_\sigma(x \otimes 1, y \otimes 1) = \langle x \otimes a, y \otimes b \rangle_h.$$

This positive-definite form is exactly what we wanted. □

Definition 1.13. We define a *Hermitian vector bundle* on S to be a finitely generated Hermitian projective R-module. When it is locally free of rank 1, we call it a *Hermitian line bundle*. Sometimes we additionally require the assumption that these are of real type, depending on the situation.

Recall that being projective over a Dedekind domain is equivalent to being flat, and also torsion-free.

Definition 1.14. We define $\widehat{\mathrm{Pic}}(S)$ to be the group of Hermitian line bundles on S up to isometry (a map $f : (E, h) \to (E', h')$ with $E \cong E'$ and $\|x\|_{h_\sigma} = \|f(x)\|_{h'_\sigma}$ for all $x \in \sigma^* E$). The group operation is still tensor product (with the natural Hermitian form induced on the tensor product $\sigma^* E \simeq \sigma^* E_1 \otimes_{\mathbf{C}} \sigma^* E_2$). The same goes for the dual line bundle. The identity is the trivial line bundle with the trivial Hermitian metric given by $\|1\|_\sigma = \|1 \otimes 1\|_\sigma = 1$ for each σ (note we can then recover the Hermitian form via the "polarization identity").

We will now define the *first arithmetic Chern class* of a Hermitian R-module. If $r > 0$ is the rank of H (at the generic point), by lifting a suitable basis of $H \otimes_R K = H \otimes_{\mathbf{Z}} \mathbf{Q}$, we may find $s_1, \dots, s_r \in H$ such that $H / \sum_{i=1}^r Rs_i$ is torsion *as a* $\mathbf{Z}$*-module* (from which it follows that $\sum_{i=1}^r Rs_i$ is free as an R-module with bais $\{s_i\}$). In particular, we may define a divisor $[H / \bigoplus_{i=1}^r Rs_i] \in Z^1(R)$ as $\sum_{\mathfrak{p} \in \mathrm{maxSpec}(R)} (\mathrm{length}_{R_\mathfrak{p}} T_\mathfrak{p})[\mathfrak{p}]$, where T is this torsion module. Note that this makes sense because a finitely generated torsion module is only supported at finitely many nonzero primes (the primes that divide an element of R that kills T), and so $T \cong \prod_{\mathfrak{p} \in \mathrm{maxSpec}(R)} T_p$ via the natural map.[1] Then

Proposition 1.15. For $s_1, \dots, s_r$ as above, define $z(s_1, \dots, s_r) \in \widehat{Z}^1(R)$ as

$$z(s_1, \dots, s_r) \coloneqq \left([H / \sum_{i=1}^r Rs_i], \sum_\sigma - \log \det \left((h_\sigma(s_i \otimes^\sigma 1, s_j \otimes_1^\sigma))_{1 \le i,j \le r} \right) [\sigma] \right).$$

This, as an element of $\widehat{CH}^1(R)$, does not depend on the choice of the s_i. We denote it by $\widehat{c}_1(\overline{H})$, and call it the *first arithmetic Chern class* of $\overline{H}$.

We note that in the case $r = 0$, i.e. when H is torsion, we can define the above construction to be $([H], 0)$.

Proof. Since we do not need the generality of this statement for now, we only give the citation [Mor14, Proposition 3.10]. Let us prove this in the special case $r = 1$, which is what we need. Let s_1' be another element of H such that H/Rs_1' is torsion as a $\mathbf{Z}$-module. If $s_1' \in Rs_1$, then choose $a \in R$ with $s_1' = as_1$, so there is an exact sequence

$$0 \to Rs_1/Rs_1' \to H/Rs_1' \to H/Rs_1 \to 0.$$

By exactness of localization and additivity of length in short exact sequences, we have $[H/Rs_1'] = [Rs_1/Rs_1'] + [H/Rs_1]$. Then $[Rs_1/Rs_1'] = [Rs_1/aRs_1] = [R/aR]$,[2] and easily

$$-\log h_\sigma(s_1' \otimes 1, s_1' \otimes 1) = -\log h_\sigma(s_1 \otimes 1, s_1 \otimes 1) - 2 \log |\sigma(a)|.$$

[1] For a proof of this, see Lemma 1.6 in [Mor14]. Summarized, because T is a finitely generated torsion module over a Dedekind domain, so only supported at finitely many maximal ideals, and in particular there is a filtration $0 \subset T_1 \subset \ldots \subset T_n = T$ where each T_i/T_{i-1} is isomorphic as R-modules to $R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$. Then the support of T is the union of the T_i, and since the claim is true for modules over the form $R/\mathfrak{p}$, write out the short exact sequences and induct on n.

[2] In higher dimensions a is replaced with $\det(A)$, and I think this is proved using modules over a PID.

Therefore we have $z(s_1') = z(s_1) + \widehat{\operatorname{div}}(\det(A)) = z(s_1) + \widehat{\operatorname{div}}(a)$.

In the general case, because H/Rs_1 is torsion over $\mathbf{Z}$, there is nonzero a such that $as_1' \in Rs_1$, so we can prove $z(as_1') = z(s_1)$ in $\widehat{CH}^1(R)$ as above, and similarly $z(as_1') = z(s_1')$ in $\widehat{CH}^1(R)$, so we are done. □

We may then define $\widehat{\deg}(H, h)$ as $\widehat{\deg}(\widehat{c}_1(H, h))$, which we also call the *arithmetic degree*. Now, note that for a torsion module $T \cong \bigoplus_{\mathfrak{p}} T_{\mathfrak{p}}$, we have $\log|T| = \sum_{\mathfrak{p}} \operatorname{length}_{R_{\mathfrak{p}}} T_{\mathfrak{p}} \log|R/\mathfrak{p}|$, because the only simple $R_{\mathfrak{p}}$-module is $R/\mathfrak{p}$. By the definition of $[T]$, we have

$$\widehat{\deg}(H, h) = \log|H/(Rs_1 + \ldots + Rs_r)| - \frac{1}{2} \sum_{\sigma} \log\det\left((h_\sigma(s_i \otimes^\sigma 1, s_j \otimes^\sigma 1))_{1 \le i,j \le r}\right), \quad (1.1)$$

or $\widehat{\deg}(H, h) = \log|H|$ when $r = 0$.

When H is projective of rank $r = 1$, i.e. a Hermitian line bundle, let s be a nonzero element of $L \otimes_R K$. For each nonzero prime $\mathfrak{p}$ of R, let $l_{\mathfrak{p}}$ be a local basis of L at $\mathfrak{p}$ (i.e. $l_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-basis of $L_{\mathfrak{p}}$, or alternatively a basis for a trivialization of L above $\mathfrak{p}$). Then there is $f_{\mathfrak{p}} \in K$ such that $s = f_{\mathfrak{p}} l_{\mathfrak{p}}$, and $\operatorname{ord}_{\mathfrak{p}}(f_{\mathfrak{p}})$ doesn't depend on the choice of $l_{\mathfrak{p}}$ as any two such choices of $l_{\mathfrak{p}}$ differ by a unit in $R_{\mathfrak{p}}$. Hence $\operatorname{ord}_{\mathfrak{p}}(s) \coloneqq \operatorname{ord}_{\mathfrak{p}}(f_{\mathfrak{p}})$ is well-defined, and we can define $\operatorname{div}(s) = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(s)[\mathfrak{p}]$. Then:

Proposition 1.16.

$$\widehat{\deg}(L, h) = \widehat{\deg}(\operatorname{div}(s), \sum_{\sigma \in K(\mathbf{C})} -\log h_\sigma(s \otimes 1, s \otimes 1)[\sigma]).$$

Here $s \otimes 1$ is an element of $(L \otimes_R K) \otimes_{K \xrightarrow{\sigma} \mathbf{C}} \mathbf{C}$.

Proof. Choose nonzero $a \in R$ such that $as \in L$. Then

$$[L/Ras] = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(as)\mathfrak{p} = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(a)\mathfrak{p} + \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(s)\mathfrak{p},$$

since by definition of $[L/Ras]$ we look for the length of $L_{\mathfrak{p}}/R_{\mathfrak{p}}as = R_{\mathfrak{p}} l_{\mathfrak{p}}/R_{\mathfrak{p}}as$ at each prime $\mathfrak{p}$, which is evidently $\operatorname{ord}_{\mathfrak{p}}(as)$ by the definition. We also have

$$\log h_\sigma(as \otimes 1, as \otimes 1) = 2\log|\sigma(a)| + \log h_\sigma(s \otimes 1, s \otimes 1),$$

so

$$([L/Ras], -\sum_{\sigma} \log h_\sigma(as \otimes 1, as \otimes 1)\sigma) = \widehat{\operatorname{div}}(a) + (\operatorname{div}(s), \sum_{\sigma \in K(\mathbf{C})} -\log h_\sigma(s \otimes 1, s \otimes 1)[\sigma]).$$

We are done by Propositions 1.15 and 1.6. □

Now, we expect from the geometric theory that $\widehat{\operatorname{Pic}}(S) \cong \widehat{CH}^1(R)$. Indeed:

Proposition 1.17. $\widehat{c}_1 : \widehat{\operatorname{Pic}}(S) \to \widehat{CH}^1(R)$ is a group isomorphism.

Proof. Note that the proof of Proposition 1.16 shows that, as an element of $\widehat{CH}^1(R)$, $\widehat{c}_1(L,h)$ can be constructed by taking *any* nonzero $s \in L \otimes K$, giving the arithmetic divisor class $\widehat{\operatorname{div}}(s) := (\operatorname{div}(s), \sum_{\sigma \in K(\mathbf{C})} -\log h_\sigma(s \otimes 1, s \otimes 1)[\sigma])$. So, to see that $\widehat{c}_1$ is a group homomorphism, given Hermitian line bundles (L,h) and (M,h'), note that the tensor product of sections l amd m of $L \otimes K$ and $M \otimes K$, as an element of $(L \otimes M) \otimes K$, has $\widehat{\operatorname{div}}(l \otimes m) = \widehat{\operatorname{div}}(l) + \widehat{\operatorname{div}}(m)$, because multiplication induces addition on valuations and logarithms.

Suppose now $\widehat{c}_1(\overline{L}) = 0$, so for a nonzero $s \in L \otimes K$, we have $\widehat{\operatorname{div}}(s)$ equaling the divisor of a rational function. In particular, if the arithmetic divisor of $a \in K^*$ equals $\widehat{\operatorname{div}}(s)$, then replace s with $a^{-1}s$ so that $\widehat{\operatorname{div}}(s) = 0$. Hence $s \in L$, because by the construction of $\widehat{\operatorname{div}}(s)$, we see that s generates $L_\mathfrak{p}$ for all $\mathfrak{p}$, so $s \in L$. Hence the map $\mathcal{O}_K \to L$ given by $1 \mapsto s$ is an isomorphism because it is so when localized at each prime, and is moreover an isometry because $\|s \otimes 1\|_\sigma = 1$ for each $\sigma \in \Sigma$. So $\overline{L}$ is the identity in $\widehat{\operatorname{Pic}}(S)$.

For surjectivity, if $(D,g) \in \widehat{Z}^1(R)$, then as in the classical case we consider $L = \mathcal{O}_S(D)$, the fractional ideal consisting of 0 and $\{a \in K^* : \operatorname{div}(a) + D \geq 0\}$. Then $L_\mathfrak{p} = l_\mathfrak{p} R_\mathfrak{p}$ where $l_\mathfrak{p} \in K$ is an element of L with valuation $-\operatorname{ord}_\mathfrak{p}(D)$ at $\mathfrak{p}$, and so $1 \in L \otimes_R K$ has $\operatorname{ord}_\mathfrak{p}(1) = -\operatorname{ord}_\mathfrak{p}(l_p) = \operatorname{ord}_\mathfrak{p}(D)$ by definition, meaning $\operatorname{div}(l) = D$. For $\sigma \in \Sigma$, we set the Hermitian metric h_σ on L_σ as given by $h_\sigma(1 \otimes 1, 1 \otimes 1) = \exp(-g_\sigma)$, so that the Hermitian line bundle (L,h) maps to (D,g) under $\widehat{c}_1$. □

Example 1.18. The degree of the trivial Hermitian line bundle R is 0. If I is an integral ideal of R with Hermitian metrics obtained by restriction from R, then $\widehat{\deg(I)} = -\log(N(I))$, because for nonzero $x \in I$,

$$\widehat{\deg(I)} = \log|I/xR| - \log \sum_\sigma \|x\|_\sigma = \log(N(xR)/N(I)) - \log(N_{K/\mathbf{Q}}(x)) = -\log(N(I)).$$

Now we return to some concepts in the geometry of numbers. In particular we would like to define some notion of "global sections" of a Hermitian line bundle that accounts for the added Archimedean data. In the geometric setting, global sections of a line bundle are exactly those admitting no poles. In other words, the valuation of such a section at each codimension-1 point of our curve/scheme is nonnegative. By analogy, for a normed finitely generated $\mathbf{Z}$-module (i.e. a finitely generated $\mathbf{Z}$-module M equipped with a norm on $M_\mathbf{R}$),

Definition 1.19. The global sections of M is the set $H^0(M) := \{x \in M : \|x\| \leq 1\}$. We also call this the set of *small sections*. We define $h^0(M) = \log|H^0(M)|$. This number is finite because M is a finite $\mathbf{Z}$-module, so if M_{tor} is the (finite) $\mathbf{Z}$-torsion submodule, then M/M_{tor}

is a $\mathbf{Z}$-lattice inside $M \otimes_{\mathbf{Z}} \mathbf{R} \cong (M/M_{tor}) \otimes_{\mathbf{Z}} \mathbf{R}$ and only has finitely many points of bounded norm. In particular $h^0(M) = h^0(M/M_{tor}) + \log|M_{tor}|$.

We can similarly define $H^0_{<1}(M)$ and $h^0_{<1}(M)$. We usually apply this definition to the case when M is a Hermitian $\mathbf{R}$-module of real type.

Remark 1.20. Note in passing that $H^0(M)$ is not even a *group*.

Definition 1.21. Let M be a finitely generated normed $\mathbf{Z}$-module. Define:

$$\widehat{\chi}(M, \|\cdot\|) = \log\left(\frac{\mathrm{vol}(B(M))}{\mathrm{vol}(M_{\mathbf{R}}/(M/M_{tor}))}\right) + \log|M_{tor}|$$

where $B(M) = B(M, \|\cdot\|) = \{x \in M_{\mathbf{R}} : \|x\| \leq 1\}$. Note that this depends on the choice of norm on M, but not on the choice of Haar measure on $M_{\mathbf{R}}$. This intuitively measures the size of $H^0(M)$: the larger it is, the smaller the fundamental mesh of M/M_{tor} is in comparison to the unit ball.

Proposition 1.22. Let (H, h) be a Hermitian $\mathbf{Z}$-module of real type (meaning we have a Hermitian metric on $H \otimes_{\mathbf{Z}} \mathbf{C}$ such that $h(x \otimes 1, y \otimes 1) \in \mathbf{R}$ for all $x, y \in H$, see Definition 1.10). Then this naturally gives a norm on $H_{\mathbf{R}}$, and

$$\widehat{\chi}(H) = \widehat{\deg}(H) + \log(V(\mathrm{rank}\ H)),$$

where $V(n)$ is the volume of the unit ball in $\mathbf{R}^n$.

Proof. It is clear from the definitions that $\widehat{\chi}(H) = \widehat{\chi}(H/H_{tor}) + \log|H_{tor}|$ and $\widehat{\deg}(H) = \widehat{\deg}(H/H_{tor}, h) + \log|H_{tor}|$, the latter since we may choose the same s_i for H and H/H_{tor} (see Proposition 1.15). So we immediately reduce to the case where H is $\mathbf{Z}$-free. Choose an orthonormal basis $x_1, \ldots, x_n$ of $H_{\mathbf{R}}$ with respect to the inner product h. Then, under the Haar measure on $H_{\mathbf{R}}$ such that the unit cube $[0,1]^n$ (with respect to this orthonormal basis) has volume 1, we have $V(n) = \mathrm{vol}(B(H)) = \mathrm{vol}(\{x \in H_{\mathbf{R}} : \|x\| \leq 1\})$.

But we will also later prove that $\mathrm{vol}(H_{\mathbf{R}}/H) = \det(h(e_i, e_j))^{1/2}$ for a $\mathbf{Z}$-basis $e_1, \ldots, e_n$ of H (see after Definition 2.2). Therefore

$$\widehat{\chi}(H) = \log\left(\frac{\mathrm{vol}(B(H))}{\mathrm{vol}(H_{\mathbf{R}}/H)}\right) = \log(\mathrm{vol}(B(H))) - \frac{1}{2}\log\det(h(e_i, e_j)) = \log(V(\mathrm{rank}\ H) + \widehat{\deg}(H, h)$$

by the formula (1.1). ∎

Example 1.23. Suppose R is the ring of integers of a number field, equipped with the norm $\|x\| := \sup_{\sigma \in K(\mathbf{C})} h^{can}_{\sigma}(x, x)^{1/2}$ as a $\mathbf{Z}$-module. Then $H^0(R)$ is the set of $x \in R$ with $\sup_{\sigma \in K(\mathbf{C})} |\sigma(x)|^2 \leq 1$. In particular $|\sigma(x)| \leq 1$ for each σ, and so $H^0(R)$ is precisely the set of roots of unity in K along with 0.

2 Arithmetic Riemann–Roch formula

In this section we would like to discuss a Riemann–Roch formula for arithmetic curves, using the h^0 we have defined above. Let's begin with something really easy, in analogy with the classical situation.

Proposition 2.1. If $\overline{L}$ is a nontrivial Hermitian line bundle with $\widehat{\deg}(\overline{L}) \leq 0$, then $h^0(\overline{L}) = 0$.

Proof. If not, then l is a nonzero element of $H^0(\overline{L})$ and by Proposition 1.16:

$$\widehat{\deg}(\overline{L}) = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(l) - \frac{1}{2} \sum_{\sigma \in \Sigma} \log h_\sigma(l \otimes 1, l \otimes 1).$$

Since $\|l\|_\sigma = h_\sigma(l \otimes 1, l \otimes 1)^{1/2} \leq 1$ for all σ, the degree must be nonnegative, so the assumption forces $\widehat{\deg}(\overline{L}) = 0$ and all terms in the sum vanish. Therefore $L = \mathcal{O}_K l$ (because l generates $L_{\mathfrak{p}}$ over each $R_{\mathfrak{p}}$) and $\|l \otimes 1\|_\sigma = 1$ for all σ, so L is isometric to the trivial line bundle. □

Now let M be a finite $\mathbf{Z}$-module equipped with a positive definite Hermitian form $\langle \cdot, \cdot \rangle : M \times M \to \mathbf{C}$. Then M has the structure of Hermitian $\mathbf{Z}$-module.

Definition 2.2. The *volume* of M (with respect to the given Hermitian form) is

$$\operatorname{vol}(M) := \exp(-\widehat{\deg}(M, \langle \cdot, \cdot \rangle)).$$

Suppose M is free with basis $e_1, \ldots, e_r$. Then the formula (1.1) shows that

$$\operatorname{vol}(M) = 0 + \exp\left(\frac{1}{2} \log \det(\langle e_i, e_j \rangle)\right) = \det(\langle e_i, e_j \rangle)^{1/2}.$$

In particular, if M is of real type, then $\operatorname{vol}(M)$ is simply the volume of the fundamental parallelogram of the lattice M in $M \otimes_{\mathbf{Z}} \mathbf{R}$.

We now return to the case where $M = R$, R is the ring of integers of a number field (again [Mor14] treats the general case of R a reduced order, which I'm not going to bother with). Recall the well-known fact

Theorem 2.3. [Neu99, I.5.2]

$$\operatorname{vol}(R, \langle \cdot, \cdot \rangle_{h^{can}}) = \sqrt{|D_K|}.$$

Here the canonical Hermitian form is given by $\langle \omega_i, \omega_j \rangle_{h^{can}} = \sum_\sigma \sigma(\omega_i)\overline{\sigma(\omega_j)}$ for $\{\omega_i\}$ forming a $\mathbf{Z}$-basis of R (Example 1.11). In particular (R, h^{can}) is the trivial Hermitian line bundle on R.

Indeed $\mathrm{vol}(R) = \det(\langle \omega_i, \omega_j \rangle_{h^{can}})^{1/2} = \det(A^T \overline{A})^{1/2} = |\det A| = \sqrt{|D_K|}$ where A is the matrix $A_{ij} = \sigma_i(\omega_j)$.

Definition 2.4. For a Hermitian R-module (H, h), define

$$\chi_{Ar}(H, h) := -\log(\mathrm{vol}(H/\mathbf{Z}, \langle \cdot, \cdot \rangle_h)) = \widehat{\deg}(H/\mathbf{Z}, \langle \cdot, \cdot \rangle_h),$$

with $H/\mathbf{Z}$ meaning H viewed as a $\mathbf{Z}$-module (Definition 2.2).

Example 2.5. From Theorem 2.3 we have $\chi_{Ar}(R, h^{can}) = -\frac{1}{2}\log|D_K|$. Next, suppose our Hermitian R-module is (I, h) where $I \subseteq R$ is an ideal and h is obtained from restricting the canonical (trivial) metric on R. Then the same computation following Theorem 2.3 for a $\mathbf{Z}$-basis of I shows that

$$\mathrm{vol}(I, h^{can}) = \sqrt{|D_K|}(R : I)$$

by looking at the change in the discriminant after scaling a basis. Hence $\chi_{Ar}(I, h) = -\frac{1}{2}\log|D_K| - \log|R/I|$.

Like $\widehat{\chi}$ before (Definition 1.21), χ_{Ar} should intuitively measure the size of $H^0(H, h)$, although this is a bit harder (for me) to see how it works right now. Note that if (H, h) is of real type, then $\|x\| = \langle x, x \rangle_h^{1/2}$ gives $H/\mathbf{Z}$ the structure of a finitely generated normed $\mathbf{Z}$-module of real type, and then

$$\widehat{\chi}(H/\mathbf{Z}, \langle \cdot, \cdot \rangle_h^{1/2}) = \widehat{\deg}(H/\mathbf{Z}, \|\cdot\|) + \log(V(\mathrm{rank}_{\mathbf{Z}} H)) = \chi_{Ar}(H, h) + \log(V(\mathrm{rank}_{\mathbf{Z}} H))$$

by Proposition 1.22.

We now come to the following theorem, where χ_{Ar} plays exactly the role of the Euler characteristic:

Theorem 2.6 (Riemann-Roch)**.** For a Hermitian R-module (H, h) of rank r, we have

$$\chi_{Ar}(H, h) = \widehat{\deg}(H, h) + r\chi_{Ar}(R, h^{can}).$$

Proof. When $r = 0$, the RHS is simply $\log|H|$, as is the LHS.So now take $r > 0$. Choose $\omega_1, \ldots, \omega_n$ a basis of R as a $\mathbf{Z}$-module, and choose $s_1, \ldots, s_r \in H$ such that $H/(Rs_1 + \ldots + Rs_r)$ is torsion as a $\mathbf{Z}$-module. Order the $\omega_i s_j$ inverse-lexicographically $\omega_1 s_1, \omega_2 s_1, \ldots$ (noting that H mod the $\mathbf{Z}$-submodule generated by all the $\omega_i s_j$ is $\mathbf{Z}$-torsion), and consider the $rn \times rn$ matrix A of inner products $(\langle \omega_i s_j, \omega_k s_l \rangle_h)$. Then the definition (1.1) gives

$$\chi_{Ar}(H, h) = \widehat{\deg}(H/\mathbf{Z}, \langle \cdot, \cdot \rangle_h) = \log|H/(Rs_1 + \ldots + Rs_r)| - \frac{1}{2}\log\det(A). \qquad (2.1)$$

We now compute the determinant of A. Divide A up into $n \times n$ matrices and let $A(j,l)$, $1 \leq j,l \leq r$, be the (j,l)-th block. Let $\sigma_1, \ldots, \sigma_n$ be the distinct embeddings of K into $\mathbf{C}$, and we have

$$\langle \omega_i s_j, \omega_k s_l \rangle_h = \sum_{m=1}^{n} h_{\sigma_m}(\omega_i s_j \otimes 1, \omega_k s_l \otimes 1) = \sum_{m=1}^{n} \sigma_m(\omega_i) h_{\sigma_m}(s_j \otimes 1, s_l \otimes 1) \overline{\sigma_m(\omega_k)},$$

and so for Δ the $n \times n$ matrix $\Delta_{ij} = \sigma_i(\omega_j)$ and $B(j,l)$ the $n \times n$ diagonal matrix with $h_{\sigma_m}(s_j \otimes 1, s_l \otimes 1)$ in the mth diagonal entry, we have

$$A(j,l) = \Delta^T B(j,l) \overline{\Delta}.$$

Hence $A = D^T B \overline{D}$ where D is an $rn \times rn$ diagonal block matrix with diagonal blocks all equal to Δ, and B is an $rn \times rn$ block matrix with blocks $B(j,l)$, $1 \leq j,l \leq r$. Hence

$$\det(A) = |\det(D)|^2 \det(B) = D_K^r \det(B) = \mathrm{vol}(R, \langle \cdot, \cdot \rangle_{h^{can}})^{2r} \det(B)$$

by Theorem 2.3. Hence from (2.1) and Definition 2.4

$$\chi_{Ar}(H,h) = \log|H/(Rs_1 + \ldots + Rs_r)| - \frac{1}{2} \log\det(B) + r\chi_{Ar}(R, h^{can}).$$

Now we have to figure out $\det(B)$. As in Proposition 1.9, consider the vector space $V := H \otimes_{\mathbf{Z}} \mathbf{C} \cong \bigoplus_{m=1}^{n} H_{\sigma_m}$ with the componentwise Hermitian inner product h_V induced from the data h. By (1.1),

$$\widehat{\deg}(H,h) = \log|H/(Rs_1 + \ldots + Rs_r)| - \frac{1}{2} \sum_{m=1}^{n} \log\det\left((h_{\sigma_m}(s_i \otimes^{\sigma_m} 1, s_j \otimes^{\sigma_m} 1))_{1 \leq i,j \leq r} \right),$$

and

$$\sum_{m=1}^{n} \log\det\left((h_{\sigma_m}(s_i \otimes 1, s_j \otimes 1))_{1 \leq i,j \leq r} \right) = \log\det(B')$$

where B' is the matrix $B'_{ij} = h_V(s_i \otimes^{\sigma_k} 1, s_j \otimes^{\sigma_l} 1)$ in the *ordered* basis $(s_i \otimes^{\sigma_k} 1)_{1 \leq k \leq n, 1 \leq i \leq r}$, the ordering being lexicographic with k first (i.e. varying s_i before σ_k; in particular B' is block-diagonal with $r \times r$ blocks). But B is, by definition, the matrix $B_{ij} = h_V(s_i \otimes^{\sigma_k} 1, s_j \otimes^{\sigma_l} 1)$ but in the *ordered* basis $(s_i \otimes^{\sigma_k} 1)_{1 \leq i \leq r, 1 \leq k \leq n}$, the ordering being lexicographic with i first (i.e. varying σ_k before s_i). Hence $\det(B) = \det(B')$ as we have only permuted the order of the basis, and so

$$\chi_{Ar}(H,h) = \log|H/(Rs_1 + \ldots + Rs_r)| - \frac{1}{2} \log\det(B') + r\chi_{Ar}(R, h^{can}) = \widehat{\deg}(H,h) + r\chi_{Ar}(R, h^{can}).$$

□

Example 2.7. Continuing the Example 2.5 of an ideal in R with the restricted trivial metric, Riemann–Roch says that

$$\chi_{Ar}(I,h) = \widehat{\deg}(I,h) + \chi_{Ar}(R,h^{can}).$$

Indeed this agrees with Examples 1.18 and 2.5, where the equation becomes $-\log|R/I| - \frac{1}{2}\log|D_K|$ on both sides (on the RHS we just split up the two terms).

We now apply the Riemann-Roch theorem to estimate the number of small sections. Let (E,h) be a Hermitian R-module. Recall Proposition 1.9 which shows that the $\mathbf{C}$-linear map $\phi : E \otimes_{\mathbf{Z}} \mathbf{C} \to \bigoplus_{\sigma \in K(\mathbf{C})} E_\sigma$, $x \otimes \alpha \mapsto (x \otimes \alpha)_\sigma$ is an isomorphism. Write ϕ_σ for the induced map $E \otimes_{\mathbf{Z}} \mathbf{C} \to E_\sigma$. For $s \in E \otimes_{\mathbf{Z}} \mathbf{R}$, also define

$$\|s\|_{sup}^{\overline{E}} = \max_{\sigma \in K(\mathbf{C})} h_\sigma(\phi_\sigma(s), \phi_\sigma(s))^{1/2} \tag{2.2}$$

(compare Example 1.23). This makes E into a normed finitely generated $\mathbf{Z}$-module. Assume now that E is torsion-free (so projective in our setting with R Dedekind, and also free as a $\mathbf{Z}$-module) and h *is of real type*. Let r_1, r_2 be the number of real and pairs of complex conjugate embeddings of K into $\mathbf{C}$. We have:

Theorem 2.8. Let r be the rank of E over R. Then

1. $$\widehat{\chi}(E, \|\cdot\|_{sup}) = \widehat{\deg}(\overline{E}) + \log\left(\frac{V(r)^{r_1}(2^r V(2r))^{r_2}}{|D_K|^{r/2}}\right),$$

   $V(n)$ being the volume of the unit ball in $\mathbf{R}^n$.

2. $$h^0(E, \|\cdot\|_{sup}) > \widehat{\deg}(\overline{E}) + \log\left(\frac{V(r)^{r_1}V(2r)^{r_2}}{2^{r(r_1+r_2)}|D_K|^{r/2}}\right).$$

   In particular if

   $$\widehat{\deg}(\overline{E}) \geq \log\left(\frac{2^{r(r_1+r_2)}|D_K|^{r/2}}{V(r)^{r_1}V(2r)^{r_2}}\right)$$

   then there is a nonzero (small) section in $H^0(\overline{E})$.

3. $$h^0_{<1}(E, \|\cdot\|_{sup}) \geq \widehat{\deg}(\overline{E}) + \log\left(\frac{V(r)^{r_1}V(2r)^{r_2}}{2^{r(r_1+r_2)}|D_R|^{r/2}}\right).$$

Proof. 1. Let $\sigma_1, \ldots, \sigma_{r_1}$ be the real embeddings and $\tau_1, \ldots, \tau_{r_2}$ be a collection consisting of one complex embedding from each conjugate pair. Let $e_{\sigma,1}, \ldots, e_{\sigma,r}$ be an orthonormal basis of E_σ with respect to the Hermitian product h_σ. We assume that these $e_{\sigma,i}$'s are chosen so that the natural anti-$\mathbf{C}$-linear map $E_\sigma \to E_{\overline{\sigma}}$ (see Proposition 1.9) sends $e_{\sigma,i}$ to $\epsilon_{\overline{\sigma},i}$. Since $\phi(E \otimes_{\mathbf{Z}} \mathbf{R})$ is precisely the subspace of $\bigoplus_{\sigma \in K(\mathbf{C})} E_\sigma$ that is invariant under the map $F_\infty : \bigoplus_{\sigma \in K(\mathbf{C})} E_\sigma \to \bigoplus_{\sigma \in K(\mathbf{C})} E_\sigma$, $(x \otimes^\sigma a) \mapsto (x \otimes^{\overline{\sigma}} \overline{a})$ (again Proposition 1.9), we see that

$$e_{\sigma_1,1} \ldots, \epsilon_{\sigma_1,r}, \ldots, e_{\sigma_{r_1},1}, \ldots, e_{\sigma_{r_1},r}, \frac{e_{\tau_1,1} + e_{\overline{\tau}_1,1}}{\sqrt{2}}, \frac{\sqrt{-1}(e_{\tau_1,1} - e_{\overline{\tau}_1,1})}{\sqrt{2}}, \ldots,$$
$$\frac{e_{\tau_{r_2},r} + e_{\overline{\tau}_{r_2},r}}{\sqrt{2}}, \frac{\sqrt{-1}(e_{\tau_{r_2},r} - e_{\overline{\tau}_{r_2},r})}{\sqrt{2}}$$

is an orthonormal $\mathbf{R}$-basis of $\phi(E \otimes_{\mathbf{Z}} \mathbf{R})$ with respect to the direct sum $\bigoplus_\sigma h_\sigma$ of the Hermitian products h_σ on $\bigoplus_{\sigma \in K(\mathbf{C})} E_\sigma$. Then we claim (for the Haar measure on $\phi(E \otimes_{\mathbf{Z}} \mathbf{R})$ coming from $\bigoplus_\sigma h_\sigma$)

$$\mathrm{vol}\{(x_\sigma) \in \phi(E \otimes_{\mathbf{Z}} \mathbf{R}) : h_\sigma(x_\sigma, x_\sigma) \leq 1 \ \forall \sigma\} = V(r)(2^r V(2r))^{r_2}.$$

Indeed, expand (x_σ) in the basis given above, with coefficients $a_{i,k}$ for $e_{\sigma_i,k}$, $b_{j,l}$ for $(e_{\tau_j,l} + e_{\overline{\tau}_j,l})/\sqrt{2}$, and $c_{j,l}$ for $(\sqrt{-1})(e_{\tau_j,l} - e_{\overline{\tau}_j,l})/\sqrt{2}$. Then

$$(x_\sigma) = \sum_{i=1}^{r_1} \sum_{k=1}^{r} a_{i,k} e_{\sigma_i,k} + \sum_{j=1}^{r_2} \sum_{l=1}^{r} \left(\frac{b_{j,l} + \sqrt{-1} c_{j,l}}{\sqrt{2}} e_{\tau_j,l} + \frac{b_{j,l} - \sqrt{-1} c_{j,l}}{\sqrt{2}} e_{\overline{\tau}_j,l} \right),$$

so the condition $h_\sigma(x_\sigma, x_\sigma) \leq 1$ for all σ is equivalent to the condition that $\sum_{k=1}^r a_{i,k}^2 \leq 1$ and $\sum_{l=1}^r b_{j,l}^2 + c_{j,l}^2 \leq 2$ for each $1 \leq i \leq r_1$ and each $1 \leq j \leq r_2$. Therefore the volume of the set of such (x_σ) is visibly $V(r)^{r_1}(2^r V(2r))^{r_2}$.

From the Definition 2.4, we have $\chi_{Ar}(E, h) = -\log(\mathrm{vol}(\phi(E \otimes_{\mathbf{Z}} \mathbf{R})/\phi(E)))$, $\mathrm{vol}(\phi(E \otimes_{\mathbf{Z}} \mathbf{R})/\phi(E))$ being the volume of the fundamental parallelogram of E in $E \otimes_{\mathbf{Z}} \mathbf{R}$ (see remark after Definition 2.2), with respect to the Hermitian product of real type $\langle x, y \rangle_h = \sum_\sigma h_\sigma(x, y)$ on E, which indeed induces the inner product $\bigoplus_\sigma h_\sigma$ on $\phi(E \otimes_{\mathbf{Z}} \mathbf{R})$ (so our choices of Haar measure on $E \otimes_{\mathbf{Z}} \mathbf{R}$ are compatible for the following claim). Then by Definition 1.21 and (2.2) we have

$$\widehat{\chi}(E, \|\cdot\|_{sup}) = \log\left(\frac{\mathrm{vol}(B(E))}{\mathrm{vol}(E \otimes_{\mathbf{Z}} \mathbf{R}/E)}\right) = \log(V(r)^{r_1}(2^r V(2r))^{r_2}) + \chi_{Ar}(E, h).$$

By Riemann-Roch (Theorem 2.6) and Example 2.5, $\chi_{Ar}(R, h^{can}) = -\log|D_K|/2$, we conclude that

$$\widehat{\chi}(E, \|\cdot\|_{sup}) = \widehat{\deg}(\overline{E}) + \log\left(\frac{V(r)^{r_1}(2^r V(2r))^{r_2}}{|D_K|^{r/2}}\right).$$

2. A standard corollary of Minkowski's Theorem [Mor14, Corollary 2.2] says that for a bounded symmetric set K and lattice Λ in a real n-dimensional vector space V, then $|K \cap \Lambda| \geq 2^{-n} \exp(\text{vol}(K)/\text{vol}(V/\Lambda))$, with strict inequality if K is furthermore closed. Taking Λ as E, V as $E \otimes_{\mathbf{Z}} \mathbf{R}$ (which has dimension rn), and K as $\{x \in V : \|x\|_{sup} \leq 1\}$, we get

$$h^0(E, \|\cdot\|_{sup}) = \log|K \cap \Lambda| > -rn \log 2 + \widehat{\chi}(E, \|\cdot\|_{sup}) = \widehat{\deg}(\overline{E}) + \log\left(\frac{V(r)^{r_1}(2^r V(2r))^{r_2}}{2^{rn}|D_K|^{r/2}}\right)$$

$$= \widehat{\deg}(E, h) + \log\left(\frac{V(r)^{r_1} V(2r)^{r_2}}{2^{r(r_1+r_2)}|D_K|^{r/2}}\right)$$

since $n = r_1 + 2r_2$ and by part (1).

3. The same argument as part (2) works, except that we instead have $K = \{x \in V : \|x\|_{sup} < 1\}$, which has the same volume as $\{x \in V : \|x\|_{sup} \leq 1\}$, and the strict inequalities must be made non-strict.

□

We now want to give an analogous definition to Definition 1.10 in the language of arithmetic divisors on R, keeping in mind the isomorphism of Proposition 1.17.

Definition 2.9. We call an arithmetic divisor $(D, g) \in \widehat{Z}^1(R)$ *conjugation invariant* if $g_\sigma = g_{\overline{\sigma}}$ for all $\sigma \in K(\mathbf{C})$.

By Definition 1.4, $\widehat{\text{div}}(a)$ for any $a \in K^*$ is conjugation invariant. Moreover, given a conjugation invariant arithmetic divisor (D, g) and considering its class in $\widehat{CH}^1(R)$, the construction in the proof of Proposition 1.17 and the previous sentence show that any Hermitian line bundle associated to (D, g) under $\widehat{c}_1$ is of real type.

Corollary 2.10. Let $(D, g) \in \widehat{Z}^1(R)$ be conjugation invariant. If $\widehat{\deg}(\overline{D}) \geq \log((2/\pi)^{r_2}\sqrt{|D_K|})$, then there is $x \in K^*$ such that $(D, g) + \widehat{\text{div}}(x) \geq 0$.

Proof. As in the proof of Proposition 1.17, construct a Hermitian line bundle of real type $(\mathcal{O}(D), h)$ from (D, g), such that $\widehat{c}_1(\mathcal{O}(D), h) = (D, g)$. From part (2) of Theorem 2.8 we see that as long as

$$\widehat{\deg}(D, g) = \widehat{\deg}(\mathcal{O}(D), h) \geq \log\left(\frac{2^{r_1+r_2}|D_K|^{1/2}}{V(1)^{r_1} V(2)^{r_2}}\right) = \log\left(\frac{2^{r_2}|D_K|^{1/2}}{\pi^{r_2}}\right),$$

then there exists a nonzero small section $x \in H^0(\mathcal{O}(D), h)$. Then by construction of $\mathcal{O}(D)$ we have $D + \text{div}(x) \geq 0$ in the finite part, and $h_\sigma(x \otimes 1, x \otimes 1) = |\sigma(x)|^2 \exp(-g_\sigma) \leq 1$ for all σ implies that $-2\log|\sigma(x)| + g_\sigma \geq 0$ for all σ, so indeed $(D, g) + \widehat{\text{div}}(x) \geq 0$. □

We may now prove the Minkowski theorem, and the finiteness of the class group using our current framework.

Theorem 2.11 (Minkowski)**.** If $[K:\mathbf{Q}] > 1$, then $|D_K| > 1$.

Proof. Pick a Hermitian line bundle over R with $\widehat{\deg}(\overline{L}) = \log((2/\pi)^{r_2}\sqrt{|D_K|})$; this can be done by picking an arithmetic divisor with specified coefficients at the archimedean places. By Corollary 2.10, we have $h^0(\overline{E}) > 0$ and therefore $\widehat{\deg}(\overline{L}) \geq 0$ by Proposition 2.1. Hence

$$\log|D_K|/2 - r_2\log(\pi/2) \geq 0.$$

Then $|D_K| \geq (\pi/2)^{2r_2}$, so we are done except when K is totally real. In the case when K is totally real (so $r_2 = 0$), we want to produce a *nontrivial* Hermitian line bundle with $\widehat{\deg}(\overline{L}) = \log((2/\pi)^{r_2}\sqrt{|D_K|})$, so that $\log((2/\pi)^{r_2}\sqrt{|D_K|}) = \log(\sqrt{|D_K|}) > 0$ by the nontriviality condition. Starting with any such Hermitian line bundle $\overline{L}$, if it were isometric to the trivial line bundle, then in terms of divisors in $\widehat{Z}^1(R)$, $\overline{L}$ would correspond to a principal arithmetic divisor. But by assumption $r_1 \geq 2$, so it is possible to re-choose the coefficients at the infinite places while maintaining the same degree, and ensuring that the new arithmetic divisor is not principal (because there are only countably many elements in K^*). ∎

Corollary 2.12. The class group $\mathrm{Cl}(K)$ is finite.

Proof. We wish to prove that the quotient of $Z^1(R) = \bigoplus_{\mathfrak{p}} \mathbf{Z}[\mathfrak{p}]$ with respect to $\{\mathrm{div}(x) : x \in K^*\}$ is finite. As in Definition 1.7, for $D = \sum_{\mathfrak{p}} n_{\mathfrak{p}}[\mathfrak{p}] \in Z^1(R)$ we define the degree to be $\sum_{\mathfrak{p}} n_{\mathfrak{p}} \log|R/\mathfrak{p}|$. Set $C_R = \log((2/\pi)^{r_2}\sqrt{|D_K|})$, and

$$\Theta = \{E \in Z^1(R) : E \geq 0, \deg(E) \leq C_R\}.$$

For $D \in Z^1(R)$, if we set

$$\overline{D} := \left(D, \sum_{\sigma} \frac{2(C_R - \deg(D))}{[K:\mathbf{Q}]}[\sigma]\right),$$

then $\widehat{\deg}(\overline{D}) = C_R$, and by Corollary 2.10 there is $x \in K^*$ with $\overline{D} + \widehat{\mathrm{div}}(x) \geq 0$ as arithmetic divisors. By the definition of $\overline{D}$ above and Definition 1.4, $\log|\sigma(a)| \leq (C_R - \deg(D))/[K:\mathbf{Q}]$ for all σ. Then by the product formula in number fields $\prod_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(x) N(\mathfrak{p}) = \prod_{\sigma} |\sigma(x)|$,

$$\deg(D + \mathrm{div}(x)) = \deg(D) + \sum_{\sigma} \log|\sigma(x)| \leq \deg(D) + \sum_{\sigma} \frac{C_R - \deg(D)}{[K:\mathbf{Q}]} = C_R,$$

and so $D + (x) \in \Theta$. But Θ is clearly a finite set and has full image under the projection $Z^1(R) \to \mathrm{Cl}(K)$. ∎

It is possible to prove the Dirichlet unit theorem via similar techniques, following the notes by Chambert–Loir.

Corollary 2.13. The abelian group $\mathcal{O}_K^*$ is finitely generated of rank $r_1 + r_2 - 1$. More specifically, the image of $\mathcal{O}_K^*$ under the logarithm map $\log : x \mapsto (\log|\sigma(x)|)_\sigma$ is a lattice inside the sum-0 hyperplane of $(\mathbf{R}^{K(\mathbf{C})})^{\mathrm{conj}}$.

Proof. Let $A = (\mathbf{R}^{K(\mathbf{C})})^{\mathrm{conj}}$ (a real vector space of dimension $r_1 + r_2$), and let $A_0 \subset A$ be the sum-0 hyperplane. Also let $\widehat{Z}^1(R)_0$ be the subgroup of degree-0 arithmetic divisors, and similarly for $\widehat{CH}^1(R)_0$. Give $\widehat{Z}^1(R)$ the topology induced by the supremum norm $\|(D,g)\| = \left\|\sum_{\mathfrak{p}} n_{\mathfrak{p}}\mathfrak{p}, \sum_\sigma g_\sigma\sigma\right\| = \max(\max_{\mathfrak{p}} n_{\mathfrak{p}}, \max_\sigma g_\sigma)$, and give all its sub/quotient groups the induced topologies.

Consider the injections $A \to \widehat{Z}^1(R)$ and $A_0 \to \widehat{Z}^1(R)_0$ given by $g \mapsto (0, g)$. Fixing an element in A such that $\widehat{\deg}(0, h) = 1$, these maps have continuous left inverses given by $(D, g) \mapsto g$ and $(D, g) \mapsto g + \widehat{\deg}(D, 0) \cdot h$. Therefore these inclusions identify A and A_0 with open (and closed) subgroups of $\widehat{Z}^1(R)$ and $\widehat{Z}^1(R)_0$, respectively. Now note that $\widehat{\mathrm{div}}(K^*) \cap A_0 = \log(\mathcal{O}_K^*)$, since the left-hand side is the subset of norm-1 elements of $\mathcal{O}_K$, i.e. exactly the units. But $\widehat{\mathrm{div}}(K^*)$ is a discrete subgroup of $\widehat{Z}^1(R)$,[3] so $\log(\mathcal{O}_K^*)$ is a discrete subgroup of A_0.

Now choose a compact subset Ω of $\widehat{Z}^1(R)_0$ that has full image under the projection to $\widehat{CH}^1(R)_0$. This can be done by noting that the set Σ of effective arithmetic divisors of degree $C_R = \log((2/\pi)^{r_2}\sqrt{|D_K|})$ is compact, because such an arithmetic divisor must have $0 \leq n_{\mathfrak{p}} \leq C_R/\log N(\mathfrak{p})$ for all the nonarchimedean coefficients (in particular $n_{\mathfrak{p}}$ must be 0 for all but finitely many $\mathfrak{p}$ of small norm), and $0 \leq g_\sigma \leq 2C_R$ for all the finitely many archimedean coefficients. The proof of Corollary 2.12 then shows that for any arithmetic divisor $\overline{D} \in \widehat{Z}^1(R)_0$ of degree 0, after translating it into Σ by adding $\overline{E} \coloneqq \left(0, \sum_\sigma \frac{2C_R}{[K:\mathbf{Q}]}[\sigma]\right)$, we see that $\overline{D}$ is equivalent (modulo a principal arithmetic divisor) to an element of $\Omega \coloneqq \Sigma - \overline{E} \subseteq \widehat{Z}^1(R)_0$. Of course, this principal arithmetic divisor need not come from a unit, and the next paragraph will rectify this when $\overline{D}$ comes from an element of A_0.

The image Θ' of Ω in $Z^1(R)$ (by forgetting the archimedean components of arithmetic divisors) is contained in Θ (defined in the proof of Corollary 2.12), so finite. For each $p \in \Theta'$ that is a principal divisor, pick $a_p \in K^*$ such that $\mathrm{div}(a_p) = p$, and let Ω' be the union of the *finitely many* $\Omega + \widehat{\mathrm{div}}(a_p)$. Because Ω is compact, Ω' is a compact subset of $\widehat{Z}^1(R)_0$. Now for $g \in A_0$, we may choose $a \in K^*$ such that $(0, g) - \widehat{\mathrm{div}}(a) \in \Omega$. The projection of this element in $Z^1(R)$ is principal $-\mathrm{div}(a) \eqqcolon p$, so $(0, g) - \widehat{\mathrm{div}}(a) + \widehat{\mathrm{div}}(a_p) \in \Omega'$. On the nonarchimedean parts, we have $\mathrm{div}(a) = \mathrm{div}(a_p)$, and hence $u \coloneqq a/a_p$ is a unit in $\mathcal{O}_K^*$.

[3] Because if $0 < t < 1$ and $\widehat{\mathrm{div}}(x)$ is in $\{x : \|x\| \leq t\}$, then we must have $\mathrm{div}(x) = 0$ and $|\sigma(x)| < t$ for all $\sigma \in K(\mathbf{C})$, so that $x \in \mathcal{O}_K$ and has monic minimal polynomial with bounded $\mathbf{Z}$-coefficients.

So $\widehat{\operatorname{div}}(u) = (0, \log(u))$ has no nonzero coefficients in the nonarchimedean part, and has $\log(u) \in A_0$. Hence $g - \log(u)$ lies in the compact subset $\Omega' \cap A_0$ of A_0, so that $A_0/\log(\mathcal{O}_K^*)$ is compact as the image of $\Omega' \cap A_0$ in this quotient is full.

Hence $\log(\mathcal{O}_K^*)$ is discrete and cocompact, so a lattice in A_0 with rank $r_1 + r_2 - 1$. The kernel of log is the roots of unity in $\mathcal{O}_K^*$ by the standard argument. □

2.1 Ampleness in the 1-dimensional case

We want to conclude with some remarks on what ampleness should mean in the arithmetic case. First, we want to determine how degree behaves under tensor product. For notation, suppose $\overline{L} = (L, h)$, $\overline{E} = (E, k)$ are Hermitian R-modules of real type. Assume moreover that (L, h) is a Hermitian *line bundle*, and the rank of E is r.

Lemma 2.14. Let $(L, h) \otimes (E, k)$ be $(L \otimes_R E, h \otimes k)$. Then

$$\widehat{c}_1((L,h) \otimes (E,k)) = r\widehat{c}_1(L,h) + \widehat{c}_1(E,k).$$

In particular (Proposition 1.15)

$$\widehat{\deg}((L,h) \otimes (E,k)) = r\widehat{\deg}(L,h) + \widehat{\deg}(E,k).$$

Proof. By the recipe of Proposition 1.15, choose $s \in L$ and $s_1, \ldots, s_r \in E$ such that L/Rs and $E/(Rs_1 + \ldots + Rs_r)$ are torsion $\mathbf{Z}$-modules. Then the $s \otimes s_i$ form a basis of $(L \otimes_R E) \otimes_R K \simeq (L \otimes_R K) \otimes_K (E \otimes_R K)$, so $(L \otimes_R E)/(R(s \otimes s_1) + \ldots + R(s \otimes s_r))$ is torsion as an R-module, hence torsion as a $\mathbf{Z}$-module. We claim that

$$[(L \otimes_R E)/(R(s \otimes s_1) + \ldots + R(s \otimes s_r))] = r[L/Rs] + [E/(Rs_1 + \ldots + Rs_r)]. \tag{2.3}$$

Indeed let ω be a local basis of $L_\mathfrak{p}$ over $R_\mathfrak{p}$ for any prime $\mathfrak{p}$ of R, and set $s = a\omega$. Since $Rs_1 + \ldots + Rs_r$ is a free R-module, we have

$$\begin{aligned}
&\operatorname{length}_{R_\mathfrak{p}} \left((L_\mathfrak{p} \otimes_{R_\mathfrak{p}} E_\mathfrak{p})/(R_\mathfrak{p}(s \otimes s_1) + \ldots + R_\mathfrak{p}(s \otimes s_r))\right) \\
= &\operatorname{length}_{R_\mathfrak{p}} \left((R_\mathfrak{p}\omega \otimes_{R_\mathfrak{p}} E_\mathfrak{p})/(R_\mathfrak{p}(a\omega \otimes s_1) + \ldots + R_\mathfrak{p}(a\omega \otimes s_r))\right) \\
= &\operatorname{length}_{R_\mathfrak{p}} \left((E_\mathfrak{p})/a(R_\mathfrak{p}s_1 + \ldots + R_\mathfrak{p}s_r)\right) \\
= &\operatorname{length}_{R_\mathfrak{p}} \left((E_\mathfrak{p})/(R_\mathfrak{p}s_1 + \ldots + R_\mathfrak{p}s_r)\right) + \operatorname{length}_{R_\mathfrak{p}} \left((R_\mathfrak{p}s_1 + \ldots + R_\mathfrak{p}s_r)/a(R_\mathfrak{p}s_1 + \ldots + R_\mathfrak{p}s_r)\right).
\end{aligned}$$

Since $R_\mathfrak{p}s_1 + \ldots + R_\mathfrak{p}s_r$ is free (of rank r) over $R_\mathfrak{p}$ with the same basis, the last term is equal to $r\operatorname{length}_{R_\mathfrak{p}}(R_\mathfrak{p}/aR_\mathfrak{p}) = r\operatorname{length}_{R_\mathfrak{p}}(L_\mathfrak{p}/sR_\mathfrak{p})$. Therefore we obtain (2.3). For the archimedean part, we have

$$\log\det((h \otimes k)_\sigma(s \otimes s_i, s \otimes s_j)) = \log\det(h_\sigma(s,s)k_\sigma(s_i,s_j)) = r\log h_\sigma(s,s) + \log\det(k_\sigma(s_i,s_j)),$$

and so the construction of $\widehat{c}_1$ gives $\widehat{c}_1((L,h) \otimes (E,k)) = r\widehat{c}_1(L,h) + \widehat{c}_1(E,k)$. □

From this we can obtain

Proposition 2.15. In the above notation, and furthermore assuming that E is projective over R (of rank r), and L and E are of real type:

1. $\widehat{\chi}(L^{\otimes n} \otimes_R E, \|\cdot\|_{sup}) = nr\widehat{\deg}(\overline{L}) + O(1)$.
2. $h^0_{<1}(L^{\otimes n} \otimes_R E, \|\cdot\|_{sup}) \geq nr\widehat{\deg}(\overline{L}) + O(1)$.
3. If $\widehat{\deg}(\overline{L}) > 0$, then for sufficiently large n, $L^{\otimes n} \otimes_R E$ is generated by elements of $\{s \in L^{\otimes n} \otimes E : \|s\|_{sup} < 1\}$.

Remark 2.16. This is a very simple case of the general *arithmetic Hilbert-Samuel formula.*

Proof. Since the rank of $L^{\otimes n} \otimes E$ is r regardless of n, parts (1) and (2) follow from Theorem 2.8 (which requires the real type hypothesis) and Proposition 2.14. For part (3) we refer to [Mor14, Proposition 3.22] as well as [Zha92, Lemma 1.6, 1.7] which develop the required concepts in the geometry of numbers for the proof. $\square$

As in the classical case, we make the following definition:

Definition 2.17. Let $\overline{L} = (L, h)$ be a Hermitian line bundle of real type. We call $\overline{L}$ *ample* if for large enough n, $L^{\otimes n}$ is generated as an R-module by elements with $\|s\|_{sup}^{\overline{L}^{\otimes n}} < 1$ for sufficiently large n.[4]

Corollary 2.18. The following are equivalent:

1. $\widehat{\deg}(\overline{L}) > 0$.
2. Let (E, k) be a Hermitian vector bundle of real type. Then for sufficiently large n, $L^{\otimes n} \otimes E$ is generated by elements s with $\|s\|_{sup} < 1$.
3. $\overline{L}$ is ample.
4. For sufficiently large n, $L^{\otimes n}$ has a nonzero section s with $\|s\|_{sup} < 1$.

Proof. (1) implies (2) by part (3) of Proposition 2.15. (2) implies (3) by taking (E, k) to be the trivial Hermitian line bundle. (3) implies (4) is trivial. (4) implies (1) because $\widehat{c}_1$ is a group homomorphism (as described in Proposition 1.17), so the degree of a Hermitian line bundle is compatible with tensor product. Then by Proposition 2.1, we see that either $\widehat{\deg}(\overline{L}) > 0$ or $\overline{L}$ is the trivial line bundle, but Example 1.23 shows that this has no nontrivial sections s with $\|s\|_{sup} < 1$. $\square$

[4] It is not clear to me why Moriwaki requires this inequality to be strict.

Example 2.19. Let $m \in \mathbf{Z}$ be a squarefree integer, $K = \mathbf{Q}(\sqrt{m})$. Set $\overline{L} = (R, \{\exp(-\lambda_i)|\sigma_i(\cdot)|\})$ for $\{\sigma_1 = \mathrm{id}, \sigma_2\} = \mathrm{Gal}(K/\mathbf{Q})$, where λ_1, λ_2 are real constants. We have

$$\widehat{\deg}(\overline{L}) = 0 - \sum_{i=1}^{2} \log(\exp(-\lambda_i)|\sigma_i(1)|^2) = \lambda_1 + \lambda_2. \tag{2.4}$$

Also, $\overline{L}^{\otimes n} = (R, \{\exp(-n\lambda_i)|\sigma_i(\cdot)|\})$. For Proposition 2.15 to be true, the condition that $\overline{L}$ is of real type is necessary.[5] For instance, suppose $m = -1, \lambda_1 < 0, \lambda_1 + \lambda_2 > 0$, so $\overline{L}$ definitely cannot be of real type. For $s = x + iy \in R$ where $x, y \in \mathbf{Z}$, $\|s\|_{sup}^{\overline{L}^{\otimes n}} \le 1$ if and only if $x^2 + y^2 \le \exp(2n\lambda_1)$, since $\lambda_1 < 0 < \lambda_2$. Hence if $n \ge 1$, then s must be 0. Therefore $\widehat{\deg}(\overline{L}) = \lambda_1 + \lambda_2 > 0$ but $\overline{L}$ cannot be ample.

Example 2.20. Take the same setup as in the previous example, but now with $m > 0$ (take $m \equiv 2, 3 \bmod 4$ for simplicity, although if we allow orders in $\mathcal{O}_K$ as Moriwaki does, we can treat $m \equiv 1 \bmod 4$ as well). In this case, all embeddings $K \hookrightarrow \mathbf{C}$ are real, so $\overline{L}$ is always of real type. Let $\lambda, \epsilon > 0$, set $\lambda_1 = \lambda + \epsilon, \lambda_2 = -\lambda$, so as in (2.4), $\widehat{\deg}(\overline{L}) = \epsilon > 0$. Then by part (3) in Proposition 2.15, for large enough n (depending on λ and ϵ of course) there is nonzero $s = x + y\sqrt{m} \in R$ with $\|s\|_{sup}^{\overline{L}^{\otimes n}} \le 1$. In particular, $|x + y\sqrt{m}| \le \exp(n(\lambda + \epsilon))$ and $|x - y\sqrt{m}| \le \exp(-n\lambda)$. Hence

$$\left|\frac{x}{y} - \sqrt{m}\right| \le \frac{1}{|y|\exp(n\lambda)}, \quad |y| \le \frac{\exp(n(\lambda + \epsilon))}{\sqrt{m}}.$$

The second inequality is true as otherwise (assuming by symmetry that $x < 0, y > 0$) $|x - y\sqrt{m}| \le \exp(-n\lambda)$ is violated. By making ϵ tend to 0, we get infinitely many x, y satisfying this Dirichlet-type approximation, albeit slightly weaker: in the standard Dirichlet theorem we should have $|x/y - \sqrt{m}| \le 1/y^2$, but here we instead have $|x/y - \sqrt{m}| \le \exp(n\epsilon)/(y^2\sqrt{m})$, which might be greater than $1/y^2$ depending on how large n is with respect to λ and ϵ.

[5] So perhaps I should have required this condition in the Definition 1.13 for Hermitian line/vector bundles, but honestly I am not sure of the correct convention.

References

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